Analytical Method for Constrained Mechanical and Structural Systems

Hee-Chang Eun*
Department of Architectural Engineering, Kangwon National University, Hyoja 2 Dong, Chuncheon, Korea
Sang-Yeol Park
Department of Civil & Environmental Engineering, Cheju National University, Jeju, Korea
Eun-Taik Lee, Heon-Soo Chung
Department of Architectural Engineering, Chung-Ang University, Seoul, Korea

The objective of this study is to present an accurate and simple method to describe the motion of constrained mechanical or structural systems. The proposed method is an elimination method to require less effort in computing Moore-Penrose inverse matrix than the generalized inverse method provided by Udwadia and Kalaba. Considering that the results by numerical integration of the derived second-order differential equation to describe constrained motion veer away the constrained trajectories, this study presents a numerical integration scheme to obtain more accurate results. Applications of holonomically or nonholonomically constrained systems illustrate the validity and effectiveness of the proposed method.

Key Words : Constraint, Control Force, Minimization, Robot, End-Effector, Generalized Inverse Matrix

1. Introduction

The motion of constrained systems is described by the simultaneous solution of unconstrained equations of motion expressed by the second-order differential equations and constraint equations by algebraic equations. It is not easy to combine them and requires a lot of efforts to describe it. Most of analytical methods depend on numerical approaches like Lagrange multiplier method. The Lagrange multiplier method relies on problem-specific approaches to the determination of the multiplier is often difficult to obtain them and hence to obtain the explicit equations of motion for systems which have a large number of degrees of freedom and many non-integrable constraints.

* Corresponding Author,
E-mail : heechang@kangwon.ac.kr
TEL : +82-33-250-6223; **FAX** : +82-33-250-6211
Department of Architectural Engineering, Kangwon National University, Hyoja 2 Dong, Chuncheon, Korea.
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The Gibbs-Appell method (Gibbs, 1879; Appell, 1899) requires a felicitous choice of quasi-coordinates and is usually amenable to problem-specific situations. This approach is likewise difficult to use, when dealing with systems having several tens of freedom and several non-integrable constraints. Kane (1983) introduced a method for nonholonomic systems based on the development of Lagrange equations from D'Alembert's Principle. Though his method is usally less tedious than the computation associated with Lagrange multipliers, it is difficult to compute vector components of acceleration. It also gets more complicated with increasing numbers of degrees of freedom. Passerello and Huston (1973) modified Kane's formulation by eliminating the computation of acceleration components. In their method, the establishment of supplementary equations to eliminate arbitrary coordinates from constraint equations may be difficult.

Udwadia and Kalaba (1992) derived an explicit equation of motion for constrained systems. They considered Gauss's Principle as the starting point and relied on developments in the field of linear algebra, specifically, the development of the concept of Moore-Penrose generalized inverse. The advantage of their method is to explicitly and simply determine the constrained motion and the constraint forces. This method requires numerical calculation of $m \times n$ generalized inverse matrix for describing the motion of n-degree-of-freedom system subjected to m constraints, $m<n$. Transforming the joint-space dynamics into the constraint-space model, You (1996), and You and Jeong (1998) presented a hybrid adaptive control law to simultaneously manipulate the end-effector position and the contact force. Park et al. (2000) presented a computational method for the motion of constrained systems by simultaneously solving the acceleration constraint equations and the equations of motion. This method considered the systems with the full rank of partial velocity matrix.

There are an infinite number of accelerations to satisfy constraint equations that are functions of acceleration components to be obtained by differentiating holonomic or nonholonomic constraints with respect to time. Constrained equation of motion is derived by selecting one of them. Hence, starting from Gauss's principle, this study derives a matrix form of constrained equation of motion. Although the proposed method is derived by the similar process such as the generalized inverse method (Udwadia and Kalaba, 1992), the proposed method takes a kind of elimination form such as other analytical methods (Gibbs, 1879; Appell, 1899; Kane, 1983; Passerello and Huston, 1973) and requires less effort in computing generalized inverse matrix than the generalized inverse method. By the elimination method, the constrained motion of n-degree-of-freedom system is described by m dynamical equations to include the effects of constraints and $(n-m)$ unconstrained equations of motion. The results by numerical integration of the derived second-order differential equation to describe constrained motion veer away constrained trajectories and exhibit the errors in the satisfaction of constraints. This study provides a numerical integration scheme to reduce the errors. Applications of holonomically or nonholonomically constrained systems illustrate the validity and effectiveness of the proposed method.

2. Equation of Constrained Motion

Consider a dynamical system described in terms of an n-vector $\mathbf{q}=[q_1\ q_2 \cdots q_n]^T$. Using the fundamental lemma of the calculus of variations, Euler-Lagrange equations are derived as follows in terms of the Lagrangian $L=T-V$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=Q_i,\ i=1,\ 2,\ \cdots,\ n \tag{1}$$

where T is the kinetic energy of the system and therefore a function of velocity $\dot{q}_i$, while V is the potential energy of the system and therefore a function of the coordinates q_i. And the Q's denote generalized forces.

Let the system be subjected to the following m constraints

$$G_j(\hat{\mathbf{q}},\ t)=0,\ j=1,\ 2,\ \cdots,\ m,\ m<n \tag{2}$$

where $\hat{\mathbf{q}}=[q_1\ \ q_2\ \ \cdots\ \ q_h]^T$, $m \leq h \leq n$. The generalized inverse method by Udwadia and Kalaba expressed the constraint equations (2) by the displacement vector space $\mathbf{q}=[q_1\ q_2 \cdots q_n]^T$. The constraint equations (2) satisfy at all times, and assuming that the constraint equations (2) are sufficiently smooth, the twice derivatives with respect to time t must also satisfy at all times.

$$\begin{aligned}\ddot{G}_j=&\sum_{l=1}^{h}\frac{\partial G_j}{\partial q_l}\ddot{q}+\sum_{l=1}^{h}\sum_{r=1}^{h}\frac{\partial}{\partial q_r}\left(\frac{\partial G_j}{\partial q_l}\right)\dot{q}_r\dot{q}_l\\&+\sum_{l=1}^{h}\frac{\partial}{\partial t}\left(\frac{\partial G_j}{\partial q_l}\right)\dot{q}_l\\&+\sum_{r=1}^{h}\frac{\partial}{\partial q_r}\left(\frac{\partial G_j}{\partial t}\right)\dot{q}_r+\frac{\partial^2 G_j}{\partial t^2}=0\end{aligned} \tag{3a}$$

$$\mathbf{A}(\hat{\mathbf{q}},\ \dot{\mathbf{q}},\ t)\ddot{\mathbf{q}}=\mathbf{b}(\hat{\mathbf{q}},\ \dot{\mathbf{q}},\ t) \tag{3b}$$

where $\mathbf{A}$ is $m \times h$ matrix and $\mathbf{b}$ is $m \times 1$ vector defined as

$$\mathbf{A}=\begin{bmatrix}\frac{\partial G_1}{\partial q_1} & \frac{\partial G_1}{\partial q_2} & \cdots & \frac{\partial G_1}{\partial q_h}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial G_m}{\partial q_1} & \frac{\partial G_m}{\partial q_2} & \cdots & \frac{\partial G_m}{\partial q_h}\end{bmatrix}_{m\times h} \tag{4a}$$

$$\mathbf{b}=-\sum_{l=1}^{h}\sum_{r=1}^{h}\frac{\partial}{\partial q_r}\left(\frac{\partial G_j}{\partial q_l}\right)\dot{q}_r\dot{q}_l -\sum_{l=1}^{h}\frac{\partial}{\partial t}\left(\frac{\partial G_j}{\partial q_l}\right)\dot{q}_l-\sum_{r=1}^{h}\frac{\partial}{\partial q_r}\left(\frac{\partial G_j}{\partial t}\right)\dot{q}_r-\frac{\partial^2 G_j}{\partial t^2} \quad (4b)$$

$$j=1,\ 2,\ \cdots,\ m$$

The rank of matrix $\mathbf{A}$ is k, which $k\leq m$.

Utilizing the fundamental theory of linear algebra, the general solution of $\mathbf{A}\ddot{\mathbf{q}}=\mathbf{b}$ with respect to $\ddot{\mathbf{q}}$ can be expressed as

$$\ddot{\mathbf{q}}=\mathbf{A}^{+}\mathbf{b}+(\mathbf{I}-\mathbf{A}^{+}\mathbf{A})\mathbf{y} \quad (5)$$

where '+' denotes the generalized inverse matrix and the vector $\mathbf{y}$ is an arbitrary vector. It can be shown that there are an infinite number of accelerations to satisfy equation (5). We need to select one of them to determine the constrained motion for the system.

The dynamic equation (1) can be also written in terms of matrix form as

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{Q}(t) \quad (6)$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the $n\times n$ positive definite mass, damping, and stiffness matrices, respectively.

Considering the acceleration vector $\ddot{\mathbf{q}}$ of equation (6) in partitioned form of h accelerations $\ddot{\mathbf{q}}_a$ corresponding to $\ddot{\mathbf{q}}$ of equation (3) and the rest $(n-h)$ accelerations $\ddot{\mathbf{q}}_b$, it can be written as

$$\begin{bmatrix}\mathbf{M}_{aa} & \mathbf{M}_{ab}\\ \mathbf{M}_{ba} & \mathbf{M}_{bb}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{q}}_a\\ \ddot{\mathbf{q}}_b\end{bmatrix}=\begin{bmatrix}\mathbf{W}_a\\ \mathbf{W}_b\end{bmatrix} \quad (7)$$

where

$$\mathbf{W}_a=-(\mathbf{C}_{aa}\dot{\mathbf{q}}+\mathbf{C}_{ab}\dot{\mathbf{q}}_b+\mathbf{K}_{aa}\mathbf{q}_a+\mathbf{K}_{ab}\mathbf{q}_b-\mathbf{Q}_a) \quad (8a)$$

$$\mathbf{W}_b=-(\mathbf{C}_{ba}\dot{\mathbf{q}}+\mathbf{C}_{bb}\dot{\mathbf{q}}_b+\mathbf{K}_{ba}\mathbf{q}_a+\mathbf{K}_{bb}\mathbf{q}_b-\mathbf{Q}_b) \quad (8b)$$

The second equation of equation (7) gives the relation

$$\ddot{\mathbf{q}}_b=-\mathbf{M}_{bb}^{-1}(\mathbf{M}_{ba}\ddot{\mathbf{q}}_a-\mathbf{W}_b) \quad (9)$$

Substituting equation (9) into the first equation of equation (7) and arranging it, we obtain

$$\mathbf{M}_a^{*}\ddot{\mathbf{q}}_a=\mathbf{R} \quad (10)$$

where

$$\mathbf{M}_a^{*}=\mathbf{M}_{aa}-\mathbf{M}_{ab}\mathbf{M}_{bb}^{-1}\mathbf{M}_{ba} \quad (11a)$$

$$\mathbf{R}=-\mathbf{M}_{ab}\mathbf{M}_{bb}^{-1}\mathbf{W}_b+\mathbf{W}_a \quad (11b)$$

By Gauss's principle, the actual accelerations of constrained systems can be obtained by the least square of the acceleration difference between $\ddot{\mathbf{q}}$ of equation (5) and $\ddot{\mathbf{q}}_a$ of equation (10) with a weighting matrix $\mathbf{M}_a^{*}$ expressed as

$$G=[\ddot{\mathbf{q}}-\ddot{\mathbf{q}}_a]^{T}\mathbf{M}_a^{*}[\ddot{\mathbf{q}}-\ddot{\mathbf{q}}_a] =\|\mathbf{M}_a^{*1/2}\ddot{\mathbf{q}}-\mathbf{M}_a^{*1/2}\ddot{\mathbf{q}}_a\|_2^2 \quad (12)$$

In order to utilize equations (3b) and (10) into equation (12), they are modified by

$$\mathbf{A}\mathbf{M}_a^{*-1/2}\mathbf{M}_a^{*1/2}\ddot{\mathbf{q}}=\mathbf{b} \quad (13a)$$

and

$$\ddot{\mathbf{q}}_a=(\mathbf{M}_a^{*})^{-1}\mathbf{R}=\mathbf{a}_a \quad (13b)$$

respectively.

The general solution of equation (13a) with respect to $\mathbf{M}_a^{*1/2}\ddot{\mathbf{q}}$ gives

$$\mathbf{M}_a^{*1/2}\ddot{\mathbf{q}}=(\mathbf{A}\mathbf{M}_a^{*-1/2})^{+}\mathbf{b} +[\mathbf{I}-(\mathbf{A}\mathbf{M}_a^{*-1/2})^{+}(\mathbf{A}\mathbf{M}_a^{*-1/2})]\mathbf{s} \quad (14)$$

where $\mathbf{s}$ is an arbitrary vector. Substitution of equations (13b) and (14) into equation (12) leads to the relation

$$\mathbf{M}_a^{*1/2}\mathbf{a}_a=(\mathbf{A}\mathbf{M}_a^{*-1/2})^{+}\mathbf{b} +[\mathbf{I}-(\mathbf{A}\mathbf{M}_a^{*-1/2})^{+}(\mathbf{A}\mathbf{M}_a^{*-1/2})]\mathbf{s} \quad (15)$$

Solving equation (15) with respect to the arbitrary vector $\mathbf{s}$, it gives

$$\mathbf{s}=[\mathbf{M}_a^{*1/2}-(\mathbf{A}\mathbf{M}_a^{*-1/2})^{+}\mathbf{A}]\mathbf{a}_a -(\mathbf{A}\mathbf{M}_a^{*-1/2})^{+}(\mathbf{A}\mathbf{M}_a^{*-1/2})\mathbf{z} \quad (16)$$

where $\mathbf{z}$ is another arbitrary vector. Substituting equation (16) into equation (14) and utilizing the fundamental properties of generalized inverse matrix, it gives

$$\mathbf{M}_a^{*1/2}\ddot{\mathbf{q}}=\mathbf{M}_a^{*1/2}\mathbf{a}_a+(\mathbf{A}\mathbf{M}_a^{*-1/2})(\mathbf{b}-\mathbf{A}\mathbf{a}_a) \quad (17)$$

Premultiplying equation (17) by $\mathbf{M}_a^{*-1/2}$, it gives

$$\ddot{\mathbf{q}}=\mathbf{a}_a+\mathbf{M}_a^{*-1/2}(\mathbf{A}\mathbf{M}_a^{*-1/2})^{+}(\mathbf{b}-\mathbf{A}\mathbf{a}_a) \quad (18)$$

Thus, the equation of motion for constrained systems can be described by solving the simultaneous solution of $(n-h)$ equation (9) and h equation (18). Premultiplying equation (18) by $\mathbf{M}_a^{*}$, the second term of the righthand side of the result defines the constraint forces, $\mathbf{F}^{c}$, expressed as

$$\mathbf{F}^c = \mathbf{M}_a^{*1/2}(\mathbf{A}\mathbf{M}_a^{*-1/2})^+(\mathbf{b}-\mathbf{A}\mathbf{a}_a) \tag{19}$$

The constrained dynamic equation (18) is composed of the sum of the the dynamical equation (13b) and the constraint force vector (19) premultiplied by $\mathbf{M}_a^{*-1}$. Equation (19) indicates that the constraint force vector is expressed as the product of the variation in acceleration trajectory due to the presence of constraints and the weighting matrix $\mathbf{M}_a^{*1/2}(\mathbf{A}\mathbf{M}_a^{*-1/2})^+$. It can be also shown that the constraint forces act in the h displacement components included in the constraint equations (2) and do not act in the $(n-h)$ displacement components. Hence, this approach is the method to eliminate the displacement components not to be included in constraints and requires less effort than the generalized inverse method to compute the constraint forces of n components. If the given constraints are functions of n displacement components, this approach exactly corresponds to the generalized inverse method.

By the similar process as the previous derivation, the constrained motion for nonholonomic systems can be obtained. Let us assume that the dynamic system expressed by equation (6) is subjected to r nonholonomic constraints

$$G_l(\dot{\check{\mathbf{q}}}, \hat{\mathbf{q}}, t)=0,\ l=1, 2, \cdots, r,\ r<n \tag{20}$$

where $\dot{\check{\mathbf{q}}}=[\dot{q}_1\ \dot{q}_2\ \cdots\ \dot{q}_o]^T$, $\hat{\mathbf{q}}=[q_1\ q_2\ \cdots\ q_p]^T$, $o\leq n$, $p\leq n$. The first derivatives of equation (20) with respect to time t can be written in form of equation (3b), where the coefficient matrices are derived as

$$\mathbf{A}=\begin{bmatrix} \frac{\partial G_1}{\partial \dot{q}_1} & \frac{\partial G_1}{\partial \dot{q}_2} & \cdots & \frac{\partial G_1}{\partial \dot{q}_o} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_l}{\partial \dot{q}_1} & \frac{\partial G_l}{\partial \dot{q}_2} & \cdots & \frac{\partial G_l}{\partial \dot{q}_o} \end{bmatrix}_{l\times o},\ \mathbf{b}=\begin{bmatrix} \frac{\partial G_1}{\partial \hat{\mathbf{q}}}\dot{\hat{\mathbf{q}}} \\ \vdots \\ \frac{\partial G_l}{\partial \hat{\mathbf{q}}}\dot{\hat{\mathbf{q}}} \end{bmatrix}_{l\times 1} \tag{21}$$

Utilizing the coefficient matrices of equation (21) and the dynamical equation (6) into equation (18), the constrained equation of motion for nonholonomic systems can be obtained. The constraint forces only act in the o velocity components included in the constraint equations (20). Hence, the proposed method describes the constrained equation by eliminating the velocity components not to be included in constraint equations and can save the computation of generalized inverse matrix. The following applications illustrate the validity of the proposed method.

3. Application I

Let us consider the dynamical motion of the system presented by Appell in 1911. The unconstrained equations of motion of the system are given by

$$m\ddot{x}=F_x,\ m\ddot{y}=F_y,\ m\ddot{z}=F_z \tag{22}$$

where F_x, F_y and F_z are the given forces. Assume that the system is subjected to a holonomic constraint given as

$$x^2+y^2=f(t) \tag{23}$$

Differentiating the constraint equation (23) twice with respect to time, it can be written in the form $\mathbf{A}\ddot{\hat{\mathbf{q}}}=\mathbf{b}$, where

$$\mathbf{A}=[2x\ 2y],\ \ddot{\hat{\mathbf{q}}}=[\ddot{x}\ \ddot{y}]^T,\ \mathbf{b}=\ddot{f}(t)-2\dot{x}^2-2\dot{y}^2 \tag{24}$$

Substituting equations (22) and (23) into equation (19) and utilizing the fundamental linear algebra, the constraint forces F_x^c and F_y^c in the x and y components are calculated as

$$\begin{bmatrix} F_x^c \\ F_y^c \end{bmatrix}=\begin{bmatrix} m^{1/2} & 0 \\ 0 & m^{1/2} \end{bmatrix}\left([2x\ 2y]\begin{bmatrix} m^{-1/2} & 0 \\ 0 & m^{-1/2} \end{bmatrix}\right)^+\left(\ddot{f}-2\dot{x}^2-2\dot{y}^2-[2x\ 2y]\begin{bmatrix} \frac{F_x}{m} \\ \frac{F_y}{m} \end{bmatrix}\right)$$
$$=\frac{1}{2x^2+2y^2}\begin{bmatrix} mx \\ my \end{bmatrix}\left(\ddot{f}-2\dot{x}^2-2\dot{y}^2-\frac{2xF_x}{m}-\frac{2yF_y}{m}\right) \tag{25}$$

Accordingly, the constrained equation of motion of the system to be composed of the third equation of equation (22) and equation (25) is written as

$$\begin{bmatrix} m\ddot{x} \\ m\ddot{y} \\ m\ddot{z} \end{bmatrix}=\begin{bmatrix} F_x+F_x^c \\ F_y+F_y^c \\ F_z \end{bmatrix} \tag{26}$$

The constraint forces by the generalized inverse method are calculated as

$$\begin{bmatrix} F_x^c \\ F_y^c \\ F_z^c \end{bmatrix} = \begin{bmatrix} m^{1/2} & 0 & 0 \\ 0 & m^{1/2} & 0 \\ 0 & 0 & m^{1/2} \end{bmatrix} \left([2x\ 2y\ 0] \begin{bmatrix} m^{-1/2} & 0 & 0 \\ 0 & m^{-1/2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^{+} \left(\ddot{f} - 2\dot{x}^2 - 2\dot{y}^2 [2x\ 2y\ 0] \begin{bmatrix} \frac{F_x}{m} \\ \frac{F_y}{m} \\ \frac{F_z}{m} \end{bmatrix} \right) \tag{27}$$

The solution of equation (27) leads to the same results as equation (25) with $F_z^c=0$. Consequently, the same equation of motion for this constrained system is obtained. Through this simple example, it is observed that the proposed method performs the reduced computation of generalized inverse matrix as the result of the elimination of displacement components.

Replacing the holonomic constraint (23), assume that the system is subjected to a nonholonomic constraint

$$\dot{x}^2+\dot{y}^2=f(t) \tag{28}$$

Differentiating equation (28) once with respect to time and substituting the result and equation (22) into equation (19), the constraint forces are calculated as

$$\begin{bmatrix} F_x^c \\ F_y^c \end{bmatrix} = \frac{1}{2\dot{x}^2+2\dot{y}^2} \begin{bmatrix} m\dot{x} \\ m\dot{y} \end{bmatrix} \left(\dot{f} - [2\dot{x}\ 2\dot{y}] \begin{bmatrix} \frac{F_x}{m} \\ \frac{F_y}{m} \end{bmatrix} \right) \tag{29}$$

and the constrained equation of motion is easily calculated by substituting equation (29) into equation (18). This application verifies that the constrained motion of nonholonomic system is efficiently obtained by the proposed method.

4. Appliction II A three-joint link robot

Consider a three-joint link robot shown in Fig. 1 moving in the XY-plane. Rigid bar 1 has length l_1, mass m_1, and moment of inertia I_1. Bar 2 has x_2, m_2, and I_2, and bar 3 has l_3, m_3, and I_3. Figure 1 shows a rotational spring and a dashpot at each joint, and shows known or unknown disturbances acting on the system. K_1, K_2, and K_3 denote spring stiffnesses, and C_1, C_2, and C_3 are damping coefficients. This system is a highly nonlinear system described by $\mathbf{q}(t)=[\theta_1\ \theta_2\ \theta_3]^T$.

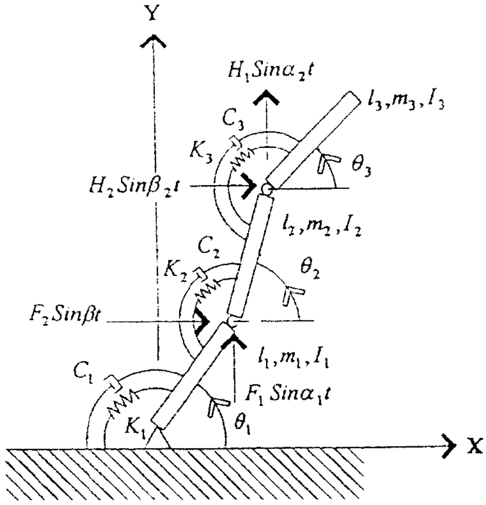


Fig. 1 A three-joint link robot

The equation of motion for this system is derived by Newtonian or Lagrangian mechanics as

$$\begin{aligned} &\begin{bmatrix} A_1 & B_1C_{12} & B_2C_{31} \\ B_1C_{12} & A_2 & B_3C_{23} \\ B_2C_{31} & B_3C_{23} & A_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 0 & B_1S_{12} & -B_2S_{31} \\ -B_1S_{12} & 0 & B_3S_{23} \\ -B_2S_{31} & B_3S_{23} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \\ \dot{\theta}_3^2 \end{bmatrix} \\ &+ \begin{bmatrix} C_1+C_2 & -C_2 & 0 \\ -C_2 & C_2+C_3 & -C_3 \\ 0 & -C_3 & C_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \\ &+ \begin{bmatrix} K_1+K_2 & -K_2 & 0 \\ -K_2 & K_2+K_3 & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \begin{bmatrix} G_1\cos\theta_1 \\ G_2\cos\theta_2 \\ G_3\cos\theta_3 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ F_3 \end{bmatrix} \end{aligned} \tag{30}$$

where

$$A_1=I_1+\left(\frac{m_1}{4}+m_2+m_1\right)l_1^2$$

$$A_2=I_2+\left(\frac{m_2}{4}+m_3\right)l_2^2,\ A_3=I_3+\frac{m_3}{4}\,l_3^2$$

$$B_1=\left(m\frac{2}{2}+m_3\right)l_1l_2$$

$$B_2=\frac{m_3}{2}\,l_1l_3,\ B_3=\frac{m_3}{2}\,l_2l_3$$

$$G_1=\left(\frac{m_1}{2}+m_2+m_3\right)gl_1$$

$$G_2=\left(\frac{m_2}{2}+m_3\right)gl_2,\ G_3=\frac{m_3}{2}\,gl_3$$

$$C_{ij}=\cos(\theta_i-\theta_j),\ S_{ij}=\sin(\theta_i-\theta_j)$$

$$P_1 = F_1 l_1 \sin \alpha_1 t \cos \theta_1 - F_2 l_1 \sin \beta_1 t \sin \theta + H_1 l_1 \sin \alpha_2 t \cos \theta_1 - H_2 l_1 \sin \beta_2 t \sin \theta_1$$
$$P_2 = H_1 l \sin \alpha_2 t \cos \theta_2 - H_2 l_2 \sin \beta_2 t \sin \theta_2,$$
$$P_3 = 0$$

Let the Cartesian coordinate of end-effector be (x_e, y_e). The coordinate is converted into Lagrangian coordinates such as

$$\begin{aligned} x_e(t) &= l_1 \cos \theta_1 + l_2 \cos \theta_2 + l_3 \cos \theta_3 \\ y_e(t) &= l_1 \sin \theta_1 + l_2 \sin \theta_2 + l_3 \sin \theta_3 \end{aligned} \tag{31}$$

Assume two constraints so that the end-effector is constrained to move in the clockwise direction along an elliptic path in the XY-plane described by the equations

$$x_e(t) - (x_e(0) + a) = a \cos(\pi - \alpha t)$$
and (32)
$$y_e(t) - y_e(0) = b \sin(\pi - \alpha t)$$

where a, b, and α are semi-major axis, semi-minor axis, and rotational speed of the end-effector about center of ellipse, respectively. Substituting equations (31) into the constraint equations (32), and differentiating them twice with respect to time t, the matrix $\mathbf{A}$ is written by

$$\mathbf{A} = \begin{bmatrix} l_1 \sin \theta_1 & l_2 \sin \theta_2 & l_3 \sin \theta_3 \\ l_1 \cos \theta_1 & l_2 \cos \theta_2 & l_3 \cos \theta_3 \end{bmatrix} \tag{33}$$

and the vector $\mathbf{b}$ is given by

$$\mathbf{b} = \begin{bmatrix} -l_1 \cos \theta_1 \dot{\theta}_1^2 - l_2 \cos \theta_2 \dot{\theta}_2^2 - l_3 \cos \theta_3 \dot{\theta}_3^2 + a\alpha^2 \cos(\pi - \alpha t) \\ -l_1 \sin \theta_1 \dot{\theta}_1^2 - l_2 \sin \theta_2 \dot{\theta}_2^2 - l_3 \sin \theta_3 \dot{\theta}_3^2 + b\alpha^2 \sin(\pi - \alpha t) \end{bmatrix} \tag{34}$$

The initial positions were selected by the values to satisfy both static equilibrium positions and the constraints.

$$\begin{aligned} &\theta_1(0) = 2.0439\ rad.,\ \theta_2(0) = 1.3862\ rad.,\ \theta_3(0) = 0.8428\ rad. \\ &\dot{\theta}_1(0) = 0.5\ rad./sec.,\ \dot{\theta}_2(0) = 1.5579\ rad./sec.,\ \dot{\theta}_3(0) = 3.2017\ rad./sec. \end{aligned} \tag{35}$$

The values of constants a, b, and α of 0.3, 0.1, and 5.0, respectively, were selected, and Table 1 gives the other values of the system. Also, assume the values of the constants in the disturbance term such as

$$\begin{aligned} &F_1 = 50\ \mathrm{N},\ F_2 = 20\ \mathrm{N},\ H_1 = 10\ \mathrm{N},\ H_2 = 100\ \mathrm{N} \\ &\alpha_1 = 3\ rad./sec.,\ \alpha_2 = 1\ rad./sec.,\ \beta_1 = 5\ rad./sec.,\ \beta_2 = 3\ rad./sec. \end{aligned} \tag{36}$$

The disturbances with these values may be large enough to excite the system.

The motion of the system is obtained by numerically integrating the second-order differential equation derived in this study. Figure 2 shows the constrained motion of the end-effector. It can be observed that the constrained motion is explicitly described through the numerical integration of the differential equation. The constrained motion is described in the state of natural equilibrium by providing the constraint forces to satisfy the given constraints. Figure 3 represents the

Table 1 Properties of the robot system

i	m_i(kg)	l_i(m)	I_i(kg·m²)	K_i(N/rad.)	C_i(N·sec/rad.)
1	30	1.1	2.5	100	6
2	30	1.1	2.5	100	6
3	10	0.5	0.13	30	3

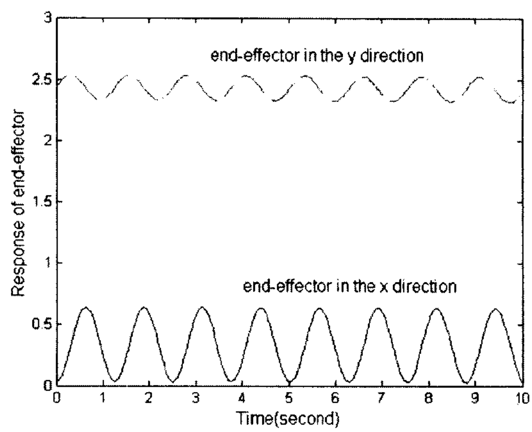


Fig. 2 Response of end-effector

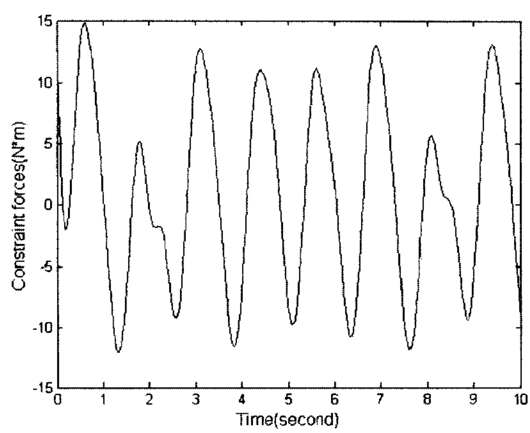


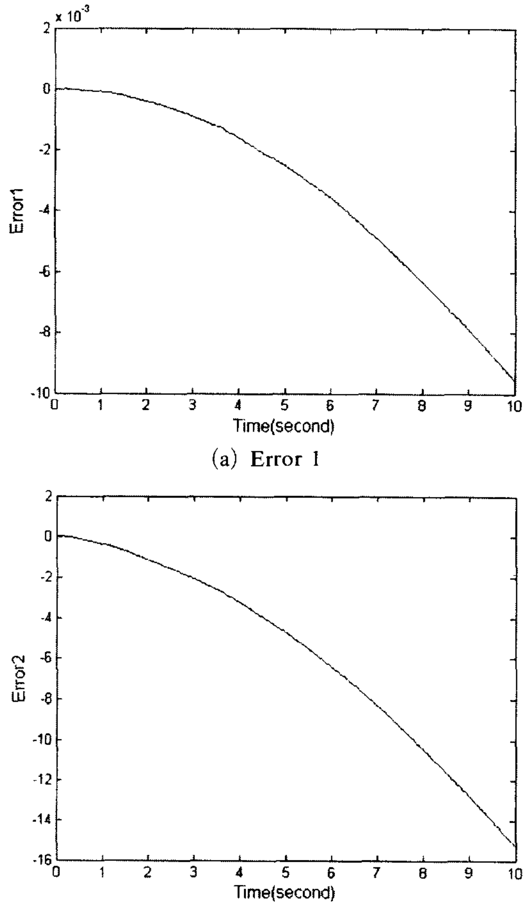
Fig. 3 Constraint forces at the joint 3

constraint forces at the joint 3 to be required for keeping the constrained motion of this system. It is verified that the constraint forces can explicitly be determined without depending on any multiplier methods.

The validity of the proposed method can be investigated by computing the errors in the satisfaction of the given constraints. Figure 4 shows the errors in the satisfaction of the constraints defined as

$$\begin{aligned} &\text{Error } 1 = x_e(t) - (x_e(0) + a) - a\cos(\pi - at) \\ &\text{Error } 2 = y_e(t) - y_e(0) - b\sin(\pi - at) \end{aligned} \tag{37}$$

The values calculated by proper integration scheme must satisfy the given constraints during numerical integration. However, the figure shows



(a) Error 1

(b) Error 2

Fig. 4 Errors in the satisfaction of the constraints

that the errors in the satisfaction of the constraints increase with time and the numerical values deviate the constrained path. Without considering any method to pull the deviated path into the constrained path, the calculated results will veer away the constrained path. Thus, a numerical integration scheme to reduce the errors is introduced in the following.

5. Numerical Integration Scheme

The second-order differential equation can be rewritten as two sets of first-order differential equations for numerical integration. The proposed second-order differential equations can be written as

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{q}}_b \end{bmatrix} = \begin{bmatrix} \xi_a(t) \\ \xi_b(t) \end{bmatrix} \tag{38a}$$

$$\begin{bmatrix} \dot{\xi}_a \\ \dot{\xi}_b \end{bmatrix} = \begin{bmatrix} \mathbf{R} + \mathbf{M}_a^{*-1/2}(\mathbf{A}\mathbf{M}_a^{*-1/2})^{+}(\mathbf{b} - \mathbf{A}\mathbf{R}) \\ -\mathbf{M}_{aa}^{-1}(\mathbf{M}_{ab}\dot{\xi}_a - \mathbf{W}_a) \end{bmatrix} \tag{38b}$$

The previous work (Eun et al., 2003) exhibited that the errors in the satisfaction of constraints come from the ignorance of the constraint equations and their first differential equations with respect to time t. The two sets of first-order differential equations were modified by inserting the first derivative of holonomic constraint (2) with respect to time or nonholonomic constraint (20) into the newly developed integration scheme. The first derivative of equation (2) with respect to time and nonholonomic constraints can be written in matrix form

$$\mathbf{D}_{m\times h}\dot{\mathbf{q}}_{h\times 1} = \mathbf{s}_{m\times 1} \tag{39a}$$

$$\mathbf{E}_{r\times o}\dot{\check{\mathbf{q}}}_{o\times 1} = \mathbf{W}_{r\times 1} \tag{39b}$$

respectively. There are an infinite number of solutions with respect to the actual velocity $\dot{\mathbf{q}}$ and $\dot{\check{\mathbf{q}}}$ of equations (39). Under the fundamental assumption that Nature chooses the minimum value of all velocities to satisfy constraints of equations (39), the least square method was utilized. Taking the least square of the difference between ξ_a of equation (38a) and, $\dot{\mathbf{q}}$ and $\dot{\check{\mathbf{q}}}$ of equations (39) with a weighting matrix $\mathbf{H}$ and following the similar procedure as the derivation of equation (18), equation (38a) for holonomic systems and equation (38b) for nonholonomic systems are developed as

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{q}}_b \end{bmatrix} = \begin{bmatrix} \xi_a(t) + \mathbf{H}^{-1/2}(\mathbf{D}\mathbf{H}^{-1/2})^+(\mathbf{s}+\mathbf{D}\xi_a) \\ \xi_b(t) \end{bmatrix} \tag{40a}$$

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{q}}_b \end{bmatrix} = \begin{bmatrix} \xi_a(t) + \mathbf{H}^{-1/2}(\mathbf{E}\mathbf{H}^{-1/2})^+(\mathbf{w}+\mathbf{E}\xi_a) \\ \xi_b(t) \end{bmatrix} \tag{40b}$$

respectively, where $\mathbf{H}$ is a positive definite matrix.

Applying this modified scheme to the three-joint link robot, the matrix $\mathbf{D}$ and vector $\mathbf{s}$ can be derived as

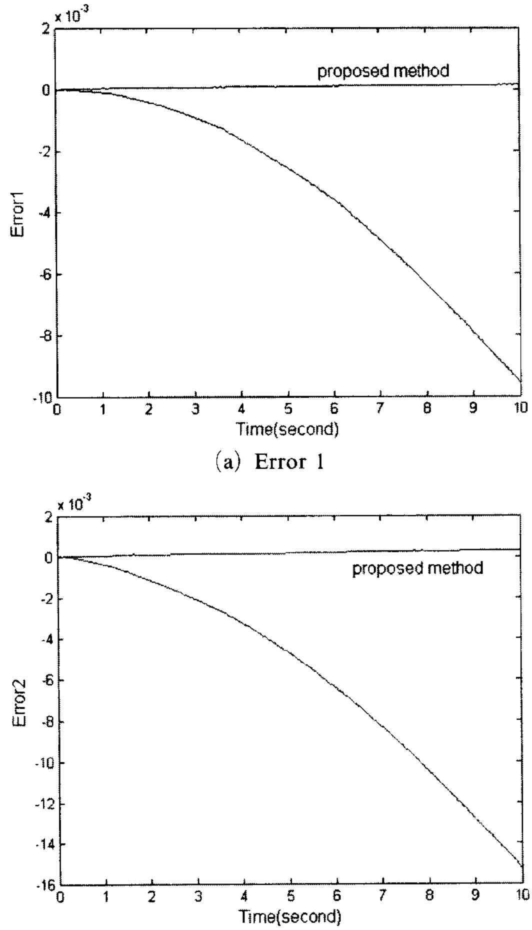
$$\mathbf{D} = \begin{bmatrix} l_1 \sin\theta_1 & l_2 \sin\theta_2 & l_3 \sin\theta_3 \\ l_1 \cos\theta_1 & l_2 \cos\theta_2 & l_3 \cos\theta_3 \end{bmatrix}$$

and (41)

$$\mathbf{s} = \begin{bmatrix} -a\alpha \sin(\pi - \alpha t) \\ -b\alpha \cos(\pi - \alpha t) \end{bmatrix}$$

respectively.

Utilizing the coefficient matrix $\mathbf{D}$ and the vector $\mathbf{s}$ of equation (41) into equation (40a), and assuming the weighting matrix $\mathbf{H}$ of a unit matrix, the modified two sets of first-order differential equations can be obtained. Taking the numerical integration to utilize MATLAB version 5.1 on a PC Pentium III, Figure 5 represents the errors in the satisfaction of the constraints. As shown by the figure, the errors are drastically reduced but do not absolutely disappeared. It is indicated that the errors in the satisfaction of constraints can be reduced by considering the first as well as second derivatives of holonomic constraint equations with respect to time in the constrained equation of motion. However, it is shown that the magnitude of the errors depends on the selection of the weighting matrix. Figure 6 exhibits the errors in the satisfaction of the constraint (32) according to three different weighting matrices: diag([1 1 1]), diag([7 3 7]), and diag([1 3 5]). The figure indicates that the magnitude of the errors is affected by the weighting matirx and it is important to properly select the weighting matrix.



(a) Error 1

(b) Error 2

Fig. 5 Errors in the satisfaction of the constraints by proposed method

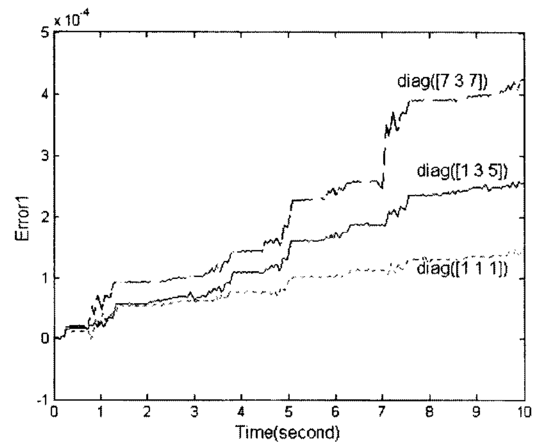


Fig. 6 Errors according to three different weighting matrices

6. Conclusions

This study presented an efficient method to describe the motion of constrained mechanical or structural systems. The proposed method took a kind of elimination form to be able to alleviate the calculation of generalized inverse matrix than the generalized inverse method provided by Udwadia and Kalaba. The errors in the satisfaction of constraints caused by numerical integration of the proposed differential equation resulted from the ignorance of the holonomic constraint equations and their first differential equations with respect to time t or nonholonomic constraints. This study proposed a numerical integration scheme to reduce the errors by inserting the first derivatives of holonomic constraint equations with respect to time or nonholonomic constraints in the constrained equation of motion. Also, it was observed that it is necessary to properly select the weighting matrix for reducing the errors and the errors are not perfectly damped out. The description of the constrained motion of several systems illustrated the validity of the proposed method.

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